Double and Iterated Integrals over Rectangles

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Overview

We defined the definite integral of a continuous function $f(x)$ over an interval $[a, b]$ as a limit of Riemann sums.

In the lecture we extend this idea to define the integral of a continuous function of two variables $f(x, y)$ over a bounded region R in the plane.

Double Integrals

In both cases the integrals are limits of approximating Riemann sums.

The Riemann sums for the integral of a single-variable function $f(x)$ are obtained by partitioning a finite interval into thin subintervals, multiplying the width of each subinterval by the value of f at a point c_k inside that subinterval, and then adding together all the products.

Double Integrals

A similar method of partitioning, multiplying, and summing is used to construct double integrals. However, this we pack a planar region R with small rectangles, rather than small subintervals. We then take the product of each small rectangle's area with the value of f at a point inside that rectangle, and finally sum together all these products.

When f is continuous, these sums converge to a single number as each of the small rectangles shrinks in both width and height. The limit is the double integral of f over R.

Double Integrals

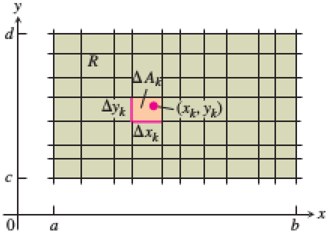
The major practical problem that arises in evaluating multiple integrals lies in determining the limits of integration.

While the intergrals of single variable were evaluated over an interval, which is determined by its two endpoints, multiple integrals are evaluated over a region in the plane or in space.

Double Integrals over Rectangles

We begin our investigation of double integrals by considering the simplest type of planar region, a rectangle. We consider a function $f(x, y)$ defined on a rectangular region R,

$$R: \quad a \leq x \leq b, \quad c \leq y \leq d.$$



The lines divide R into n rectangular pieces, where the number of such pieces n gets large as the width and height of each piece gets small.

Double Integrals over Rectangles

These rectangles form a **partition** of R. A small rectangular piece of width Δx and height Δy has area

$$\Delta A = \Delta x \, \Delta y.$$

If we number the small pieces partitioning R in some order, then their areas are given by numbers

$$\Delta A_1, \ \Delta A_2, \ \ldots, \ \Delta A_n$$

where ΔA_k is the area of the kth small rectangle.

Double Integrals over Rectangles

To form a Riemann sum over R, we choose a point (x_k, y_k) in the kth small rectangle, multiply the value of f at that point by the area ΔA_k, and add together the products

$$S_n = \sum_{k=1}^{n} f(x_k, y_k)\Delta A_k.$$

Depending on how we pick (x_k, y_k) in the kth small rectangle, we may get different values for S_n.

Double Integrals over Rectangles

We are interested in what happens to these Riemann sums as the widths and heights of all the small rectangles in the paritition of R approach zero.

The **norm** of a partition P, written $\|P\|$, is the largest width or height of any rectangle in the partition.

Double Integrals over Rectangles

Sometimes the Riemann sums converge as the norm of P goes to zero, written $\|P\| \to 0$. The resulting limit is then written as

$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} f(x_k, y_k)\Delta A_k.$$

As $\|P\| \to 0$ and the rectangles get narrow and short, their number n increases, so we can also write this limit as

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k, y_k)\Delta A_k$$

with the understanding that $\Delta A_k \to 0$ as $n \to \infty$ and $\|P\| \to 0$.

There are many choices involved in a limit of this kind. The collection of small rectangles is determined by the grid of vertical and horizontal lines that determine a rectangular partition of R. In each of the resulting small rectangles there is a choice of an arbitrary point (x_k, y_k) at which f is evaluated. These choices together determine a single Riemann sum.

To form a limit, we repeat the whole process again and again, choosing partitions whose rectangle widths and heights both go to zero and whose number goes to infinity.

Double Integrals over Rectangles

When a limit of the sums S_n exists, giving the same limiting value no matter what choices are made, then the function f is said to be **integrable** and the limit is called a **double integral** of f over R, written as

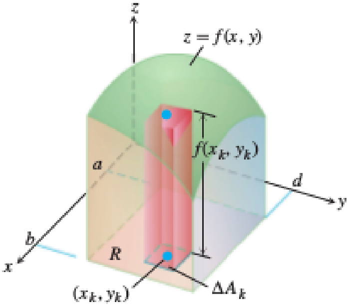
$$\iint\limits_R f(x,y)\,dA \quad \text{or} \quad \iint\limits_R f(x,y)\,dx\,dy.$$

It can be shown that if $f(x,y)$ is continuous function throughtout R, then f is integrable as in the single-variable case discussed earlier.

Many discontinuous functions are also integrable, including functions which are discontinuous only on a finite number of points or smooth curves.

Double Integrals as Volumes

When $f(x, y)$ is a positive function over a rectangular region R in the xy-plane, we may interpret the double integral of f over R as the volume of the 3-dimensional solid region over the xy-plane bounded below by R and above by the surface $z = f(x, y)$.



Double Integrals as Volumes

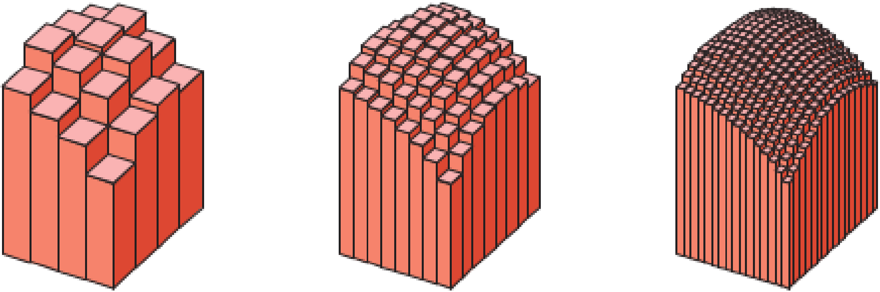
Each term $f(x_k, y_k)\Delta A_k$ in the sum $S_n = \sum f(x_k, y_k)\Delta A_k$ is the volume of a vertical rectangular box that approximates the volume of the portion of the solid that stands directly above the base ΔA_k.

The sum S_n thus approximates what we want to call the total volume of the solid. We define this volume to be

$$\text{Volume} = \lim_{n\to\infty} S_n = \iint_R f(x,y)\ dA$$

where $\Delta A_k \to 0$ as $n \to \infty$.

The following figure shows Riemann sum approximations to the volume becoming more accurate as the number n of boxes increases.

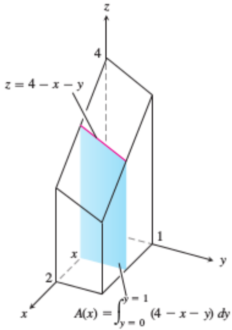


(a) $n = 16$ (b) $n = 64$ (c) $n = 256$

Suppose that we wish to calculate the volume under the plane $z = 4 - x - y$ over the rectangular region

$$R : 0 \leq x \leq 2, \quad 0 \leq y \leq 1$$

in the xy-plane.



Fibini's Theorem for Calculating Double Integrals

If we apply the method of slicing, with slices perpendicular to the x-axis, then the volume is

$$\int_{x=0}^{x=2} A(x)\, dx$$

where $A(x)$ is the cross-sectional area at x.

For each value of x, we may calculate $A(x)$ as the integral

$$\int_{y=0}^{y=1} (4 - x - y)\, dy$$

which is the area under the curve $z = 4 - x - y$ in the plane of the cross-section at x.

Fibini's Theorem for Calculating Double Integrals

In calculating $A(x)$, x is held fixed and the integration takes place with respect to y.

Hence the volume of the entire solid is

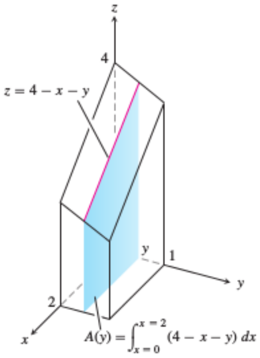
$$\text{Volume} = \int_{x=0}^{x=2} A(x)\ dx = \int_{x=0}^{x=2} \int_{y=0}^{y=1} (4 - x - y)\ dx\ dy.$$

Fibini's Theorem for Calculating Double Integrals

The expression on the right, called an **iterated** or **repeated integral**, says that the volume is obtained by integrating $4 - x - y$ with respect to y from $y = 0$ to $y = 1$, holding x fixed, and then integrating the resulting expression in x with respect to x from $x = 0$ to $x = 2$.

The limits of integration 0 and 1 are associated with y, so they are placed on the integral closest to dy. The other limits of integration, 0 and 2, are associated with the variable x, so they are placed on the outside integral symbol that is paired with dx.

What would have happened if we had calculated the volume by slicing with planes perpendicular to the y-axis?



As a function of y, the typical cross-sectional is shown above. Hence the volume of the entire solid is

$$\text{Volume} = \int_{y=0}^{y=1} \int_{x=0}^{x=2} (4 - x - y)\ dy\ dx.$$

Do both iterated integrals give the value of the double integral?

The answer is 'yes', since the double integral measures the volume of the same region as the two iterated integrals.

Fubini's Theorem (First Form)

A theorem published in 1907 by Guido Fubini says that the double integral of any continuous function over a rectangle can be calculated as an iterated integral in either order of integration.

Theorem 1.

If $f(x,y)$ is continuous throughtout the rectangular region $R : a \leq x \leq b,\ c \leq y \leq d$, then

$$\iint\limits_R f(x,y)\ dA = \int_c^d \int_a^b f(x,y)\ dx\ dy = \int_a^b \int_c^d f(x,y)\ dy\ dx.$$

Fubini's Theorem (First Form)

Fubini's Theorem says that double integrals over rectangles can be calculated as iterated integrals. Thus, we can evaluate a double integral by integrating with respect to one variable at a time.

Fubini's Theorem also says that we may calculate the double integral by integrating in either order, a genuine convenience. When we calculate a volume by slicing, we may use either planes perpendicular to the x-axis or planes perpendicular to the y-axis.

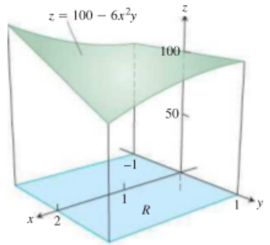
Example

Example 2.

Calculate $\iint_R f(x,y)\,dA$ *for*

$$f(x,y) = 100 - 6x^2 y \qquad \textit{and} \qquad R: \quad 0 \le x \le 2, \quad -1 \le y \le 1.$$

Solution



The above figure displays the volume beneath the surface. By Fubini's Theorem,

$$\iint\limits_R f(x,y)\,dA = \int_{-1}^{1}\int_{0}^{2}(100-6x^2y)\,dx\,dy = \int_{-1}^{1}\left[100x-2x^3y\right]_{x=0}^{x=2}\,dy = 400.$$

Reversing the order of integration gives the same answer.

$$\int_0^2\int_{-1}^{1}(100-6x^2y)\,dy\,dx = 400.$$

The double integral $\iint_R f(x,y)\,dA$ gives the volume under this surface over the rectangular region R.

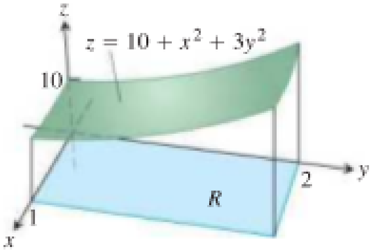
Example 3.

Find the volume of the region bounded above by the ellipitical paraboloid

$$z = 10 + x^2 + 3y^2$$

and below by the rectangle $R : 0 \leq x \leq 1,\ 0 \leq y \leq 2.$

Solution



The surface and volume are shown in the figure. The volume is given by the double integral

$$V = \iint_R (10 + x^2 + 3y^2)\, dA = \int_0^1 \int_0^2 (10 + x^2 + 3y^2)\, dy\, dx = \frac{86}{3}.$$

The double integral $\iint_R f(x, y)\, dA$ gives the volume under this surface over the rectangular region R.

Exercises 4.

Evaluate the following iterated integrals

1. $\displaystyle\int_1^2 \int_0^4 2xy \, dy \, dx$
2. $\displaystyle\int_0^1 \int_0^1 \left(1 - \frac{x^2 + y^2}{2}\right) dx \, dy$
3. $\displaystyle\int_1^4 \int_0^4 \left(\frac{x}{2} + \sqrt{y}\right) dx \, dy$
4. $\displaystyle\int_0^{\ln 2} \int_1^{\ln 5} e^{2x+y} \, dy \, dx$
5. $\displaystyle\int_\pi^{2\pi} \int_0^{\pi} (\sin x + \cos y) \, dx \, dy$

Solutions

1. $\int_1^2 \int_0^4 2xy\ dy\ dx = \int_1^2 \left[xy^2\right]_0^4 dx = \int_1^2 16x dx = 24$
2. $\int_0^1 \int_0^1 \left(1 - \frac{x^2+y^2}{2}\right) dx\ dy = \int_0^1 \left[x - \frac{x^3}{6} - \frac{xy^2}{2}\right]_0^1 dy = \int_0^1 \left(\frac{5}{6} - \frac{y^2}{2}\right) dy = \frac{2}{3}$
3. $\int_1^4 \int_0^4 \left(\frac{x}{2} + \sqrt{y}\right) dx\ dy = \int_1^4 \left[\frac{1}{4}x^2 + x\sqrt{y}\right]_0^4 dy = \int_1^4 (4 + 4y^{1/2}) dy = \frac{92}{3}$
4. $\int_0^{\ln 2} \int_1^{\ln 5} e^{2x+y} dy\ dx = \int_0^{\ln 2} \left[e^{2x+y}\right]_1^{\ln 5} dx = \int_0^{\ln 2} (5e^{2x} - e^{2x+1}) dx = \frac{3}{2}(5 - e)$
5. $\int_\pi^{2\pi} \int_0^\pi (\sin\ x + \cos\ y) dx\ dy = \int_\pi^{2\pi} \left[-\cos\ x + x \cos\ y\right]_0^\pi dy = \int_\pi^{2\pi} (2 + \pi \cos\ y) dy = 2\pi$

Exercises 5.

Evaluate the following double integrals over the given regions R.

1. $$\iint_R \left(\frac{\sqrt{x}}{y^2}\right) dA, \quad R:\ 0 \le x \le 4, \quad 1 \le y \le 2$$
2. $$\iint_R e^{x-y}\, dA, \quad R:\ 0 \le x \le \ln 2, \quad 0 \le y \le \ln 2$$
3. $$\iint_R xye^{xy^2}\, dA, \quad R:\ 0 \le x \le 2, \quad 0 \le y \le 1$$
4. $$\iint_R \frac{y}{x^2y^2+1}\, dA, \quad R:\ 0 \le x \le 1, \quad 0 \le y \le 1$$

Solutions

1. $\displaystyle\iint_R \frac{\sqrt{x}}{y^2}\,dA = \int_0^4\int_1^2 \frac{\sqrt{x}}{y^2}dy\ dx = \int_0^4 \left[-\frac{\sqrt{x}}{y}\right]_1^2 dx = \int_0^4 \frac{1}{2}x^{1/2}dx = \left[\frac{1}{3}x^{3/2}\right]_0^4 = \frac{8}{3}$
2. $\displaystyle\iint_R e^{x-y}\,dA = \int_0^{\ln 2}\int_0^{\ln 2} e^{x-y}dy\ dx = \int_0^{\ln 2}[-e^{x-y}]_0^{\ln 2}dx = \int_0^{\ln 2}(-e^{-x-\ln 2}+e^x)dx =$ $\displaystyle\left[-e^{x-\ln 2}+e^x\right]_0^{\ln 2} = \frac{1}{2}$
3. $\displaystyle\iint_R x\ y\ e^{x\ y^2}\,dA = \int_0^2\int_0^1 x\ y\ e^{xy^2}dy\ dx = \int_0^2 \left[\frac{1}{2}e^{xy^2}\right]_0^1 dx = \int_0^2 \left(\frac{1}{2}e^x - \frac{1}{2}\right)dx =$ $\displaystyle\left[\frac{1}{2}e^x - \frac{1}{2}x\right]_0^2 = \frac{1}{2}(e^2-3)$
4. $\displaystyle\iint_R \frac{y}{x^2y^2+1}\,dA = \int_0^1\int_0^1 \frac{y}{(xy)^2+1}dx\ dy = \int_0^1 \left[\tan^{-1}(x\ y)\right]_0^1 dy = \int_0^1 \tan^{-1} y\ dy =$ $\displaystyle\left[y\tan^{-1}y - \frac{1}{2}\ln|1+y^2|\right]_0^1 = \frac{\pi}{4} - \frac{1}{2}\ln 2$

Integrate f over the given region.

Exercises 6.

1. Square : $f(x,y) = 1/(xy)$ *over the square* $1 \leq x \leq 2$, $1 \leq y \leq 2$
2. Rectangle : $f(x,y) = y \cos xy$ *over the rectangle* $0 \leq x \leq \pi$, $0 \leq y \leq 1$

$$z = f(x,y)$$

Solutions

1. $$\int_1^2 \int_1^2 \frac{1}{xy} dy\ dx = \int_1^2 \frac{1}{x}(\ln 2 - \ln\ 1) dx = (\ln 2) \int_1^2 \frac{1}{x} dx = (\ln\ 2)^2$$
2. $$\int_0^1 \int_0^\pi y\ cos\ xy\ dx\ dy = \int_0^1 \Big[\sin\ xy\Big]_0^\pi dy = \int_0^1 \sin\ \pi y\ dy = \Big[-\frac{1}{\pi}\cos\ \pi y\Big]_0^1 = -\frac{1}{\pi}(-1-1) = \frac{2}{\pi}$$

Exercises 7.

1. *Find the volume of the region bounded above by the paraboloid* $z = x^2 + y^2$ *and below by the square* $R : -1 \leq x \leq 1$, $-1 \leq y \leq 1$.
2. *Find the volume of the region bounded above by the ellipitical paraboloid* $z = 16 - x^2 - y^2$ *and below by the square* $R : 0 \leq x \leq 2$, $0 \leq y \leq 2$.
3. *Find the volume of the region bounded above by the plane* $z = 2 - x - y$ *and below by the square* $R : 0 \leq x \leq 1$, $0 \leq y \leq 1$.
4. *Find the volume of the region bounded above by the plane* $z = y/2$ *and below by the rectangle* $R : 0 \leq x \leq 4$, $0 \leq y \leq 2$.
5. *Find the volume of the region bounded above by the surface* $z = 2 \sin x \cos y$ *and below by the rectangle* $R : 0 \leq x \leq \pi/2$, $0 \leq y \leq \pi/4$.
6. *Find the volume of the region bounded above by the surface* $z = 4 - y^2$ *and below by the rectangle* $R : 0 \leq x \leq 1$, $0 \leq y \leq 2$.

Solutions

1. $V = \iint_R f(x,y)dA = \int_{-1}^{1}\int_{-1}^{1}(x^2+y^2)dy\ dx = \int_{-1}^{1}[x^2y+\frac{1}{3}y^3]_{-1}^{1}dx = \int_{-1}^{1}(2x^2+\frac{2}{3})dx = \left[\frac{2}{3}x^3+\frac{2}{3}x\right]_{-1}^{1} = \frac{8}{3}$
2. $V = \iint_R f(x,y)dA = \int_0^2\int_0^2(16-x^2-y^2)dy\ dx = \int_0^2[16y-x^2y-\frac{1}{3}y^3]_0^2dx = \int_0^2\left(\frac{88}{3}-2x^2\right)dx = \left[\frac{88}{3}x-\frac{2}{3}x^3\right]_0^2 = \frac{160}{3}$
3. $V = \iint_R f(x,y)dA = \int_0^1\int_0^1(2-x-y)dy\ dx = \int_0^1\left[2y-xy-\frac{1}{2}y^2\right]_0^1dx = \int_0^1\left(\frac{3}{2}-x\right)dx = \left[\frac{3}{2}x-\frac{2}{2}x^2\right]_0^1 = 1$
4. $V = \iint_R f(x,y)dA = \int_0^4\int_0^2\frac{y}{2}dy\ dx = \int_0^4\left[\frac{y^2}{4}\right]_0^2dx = \int_0^4 1\ dx = [x]_0^4 = 4$
5. $V = \iint_R f(x,y)dA = \int_0^{x/2}\int_0^{x/4}2\sin\ x\ \cos\ y\ dy\ dx = \int_0^{x/2}[2\sin\ x\ \sin\ y]_0^{x/4}dx = \int_0^{x/2}(\sqrt{2}\sin x)dx = [-\sqrt{2}\cos\ x]_0^{x/2} = \sqrt{2}$
6. $V = \iint_R f(x,y)dA = \int_0^1\int_0^2(4-y^2)dy\ dx = \int_0^1\left[4y-\frac{1}{3}y^3\right]_0^2dx = \int_0^1\left(\frac{16}{3}\right)dx = \left[\frac{16}{3}x\right]_0^1 = \frac{16}{3}$

References

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